

# Complexity Bounds for MCMC via Diffusion Limits

by

Gareth O. Roberts<sup>1</sup> and Jeffrey S. Rosenthal<sup>2</sup>

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**Abstract.** We connect known results about diffusion limits of Markov chain Monte Carlo (MCMC) algorithms to the Computer Science notion of algorithm complexity. Our main result states that any diffusion limit of a Markov process implies a corresponding complexity bound (in an appropriate metric). We then combine this result with previously-known MCMC diffusion limit results to prove that under appropriate assumptions, the Random-Walk Metropolis (RWM) algorithm in  $d$  dimensions takes  $O(d)$  iterations to converge to stationarity, while the Metropolis-Adjusted Langevin Algorithm (MALA) takes  $O(d^{1/3})$  iterations to converge to stationarity.

## 1. Introduction.

In the computer science literature, algorithms are often analysed in terms of “complexity” bounds. In the Markov chain Monte Carlo (MCMC) literature, algorithms are sometimes understood in terms of diffusion limits. The purpose of this note is to connect these two approaches, and in particular to show that diffusion limits sometimes imply complexity bounds.

Complexity results in computer science go back at least to Cobham (1964), and took on greater focus with the pioneering *NP-complete* work of Cook (1971). In the Markov chain context, computer scientists have been bounding convergence times of Markov chain algorithms since at least Jerrum and Sinclair (1989), focusing largely on spectral gap bounds for Markov chains on finite state spaces. More recently, attention has turned to bounding spectral gaps of modern Markov chain algorithms on general (e.g. uncountable) state spaces, again primarily via spectral gaps (e.g. Woodard et al., 2009a, 2009b). These bounds often

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<sup>1</sup>Department of Statistics, University of Warwick, CV4 7AL, Coventry, U.K. Email: [g.o.roberts@lancaster.ac.uk](mailto:g.o.roberts@lancaster.ac.uk). Supported in part by EPSRC grants EP/20620/01 and EP/S61577/01.

<sup>2</sup>Department of Statistics, University of Toronto, Toronto, Ontario, Canada M5S 3G3. Email: [jeff@math.toronto.edu](mailto:jeff@math.toronto.edu). Web: <http://probability.ca/jeff/> Supported in part by NSERC of Canada.

focus on the order of the convergence time in terms of some parameter such as the dimension  $d$  of the corresponding state space.

Meanwhile, in statistics, MCMC algorithms are extremely widely used and studied (see e.g. Brooks et al., 2011, and the many references therein), and their running times are an extremely important practical issue. They have been studied from a variety of perspectives, including directly bounding the convergence in total variation distance (see e.g. Rosenthal, 1995b, 1996, 2002; Jones and Hobert, 2001, 2004; and references therein), convergence “diagnostics” via statistical analysis of the Markov chain output (e.g. Gelman and Rubin, 1992), and most notably by proving weak convergence limits of sped-up versions of the algorithms to diffusion limits (e.g. Roberts et al., 1997; Roberts and Rosenthal, 1998).

The MCMC direct total variation bounds are sometimes presented in terms of the convergence order (e.g. see Rosenthal, 1995a, for order bounds for a Gibbs sampler for a variance components model). In addition, the MCMC diffusion limits often involve speeding up the original algorithm by a certain order, and then proving weak convergence to a fixed process which converges in  $O(1)$  iterations, thus giving them the flavour of complexity order bounds too. However, the MCMC results are typically not stated precisely in terms of convergence time complexity results, and (perhaps because of this) they are often overlooked by the computer science complexity community.

In this paper, we attempt to connect these two streams of Markov chain convergence time bounds. In particular, we establish (Theorem 1) that results about diffusion limits do directly imply corresponding complexity bounds (using an appropriate convergence metric as described below). We then apply our theorem to previous results about diffusion limits of MCMC algorithms (Section 3), to establish running time complexity order bounds for such MCMC algorithms as the Random-Walk Metropolis algorithm (Theorem 2) and the Metropolis-adjusted Langevin algorithm (Theorem 3).

## 2. Assumptions and Main Result.

Let  $(\mathcal{X}, \mathcal{F}, \rho)$  be a general measurable metric space, i.e. a non-empty (and possibly uncountable) set  $\mathcal{X}$  endowed with a metric  $\rho$  which induces a Borel  $\sigma$ -algebra  $\mathcal{F}$  of measurable subsets. We wish to bound the convergence of a stochastic process  $\{X_t\}$  on  $(\mathcal{X}, \mathcal{F})$  to its stationary probability distribution  $\pi$ . To measure the distance to stationarity, on finite state spaces one often (see e.g. Aldous and Fill, 2002, Section 2.4.1) uses the total variation distance defined by

$$\|\mathcal{L}_x(X_t) - \pi\|_{TV} := \sup_{|f| \leq 1} \left| \mathbf{E}_x[f(X_t)] - \pi(f) \right|$$

where the supremum is taken over all measurable functions  $f : \mathcal{X} \rightarrow \mathbf{R}$  with  $|f(x)| \leq 1$  for all  $x \in \mathcal{X}$ . Here  $\mathcal{L}_x(X_t)$  is the law of  $X_t$  conditional on starting at  $X_0 = x$ , and  $\mathbf{E}_x[f(X_t)]$  is the expected value of  $f$  with respect to this law, and  $\pi(f) = \int f(x) \pi(dx)$  is the expected value of  $f$  with respect to  $\pi$ .

This total variation distance can also be used on general state spaces in many instances (see e.g. Rosenthal, 1995). However, it is not appropriate for bounding the weak convergence which arises in the diffusion context, since it may not go to zero for processes which converge only weakly to stationarity, so we do not use it here. Instead, we let

$$\text{Lip}_1^1 = \left\{ f : \mathcal{X} \rightarrow \mathbf{R}, |f(x) - f(y)| \leq \rho(x, y) \ \forall x, y \in \mathcal{X}, |f| \leq 1 \right\}$$

be the set of all functions from  $\mathcal{X}$  to  $\mathbf{R}$  with Lipschitz constant  $\leq 1$  and with  $|f(x)| \leq 1$  for all  $x \in \mathcal{X}$ , and use the distance function

$$\|\mathcal{L}_x(X_t) - \pi\|_{KR} := \sup_{f \in \text{Lip}_1^1} \left| \mathbf{E}_x[f(X_t)] - \pi(f) \right|.$$

(Here “KR” stands for “Kantorovich-Rubinstein”; see the proof of Proposition 6 below.) The distance  $\|\cdots\|_{KR}$  is similar to, but more restrictive than, the total variation distance, and we will see below (Proposition 6) that it metrises weak convergence and so is appropriate for our purposes.

We also note that many approaches to stationary instead directly bound the spectral gap of the corresponding Markov operator (e.g. Woodard et al., 2009b). However, on general state spaces, the spectral gap is zero for Markov chains which are not “geometrically ergodic” (see e.g. Theorem 2 of Roberts and Rosenthal, 1997). Furthermore, many MCMC algorithms are not geometrically ergodic (e.g. the Random-Walk Metropolis algorithm on target distributions with heavier-than-exponential tails, see Theorem 3.3 of Mengersen and Tweedie, 1996). They also are often not reversible, which makes spectral gaps harder to study or interpret. For these reasons, we do not wish to restrict attention to spectral gaps, which is another reason that we use the metric  $\|\cdots\|_{KR}$ .

A related issue is what initial states  $X_0$  should be considered. On finite state spaces, one often (e.g. Jerrum and Sinclair, 1989, Section 2) considers the worst case, by taking supremum over all initial states  $x$ , i.e. uses something like  $\sup_{x \in \mathcal{X}} \|\mathcal{L}_x(X_t) - \pi\|_{TV}$ . But this supremum is also frequently inappropriate on general state spaces. For instance, if  $\mathcal{X}$  is unbounded, then as  $t$  increases one can start from worse and worse states  $X_0$  so that the supremum will never go to 0. Instead, we need to specify more precisely which initial state(s)  $X_0$  to consider. As a concrete choice, we will take the  $\pi$ -average of the distances to stationarity from all initial states  $X_0$  in  $\mathcal{X}$ . That is, for any Markov chain  $\{X_t\}$  on  $(\mathcal{X}, \mathcal{F})$  with stationary distribution  $\pi$ , we measure the distance to stationarity at time  $t$  by the distance function

$$\mathbf{E}_{X_0 \sim \pi} \|\mathcal{L}_{X_0}(X_t) - \pi\|_{KR} := \int_{x \in \mathcal{X}} \pi(dx) \|\mathcal{L}_x(X_t) - \pi\|_{KR}.$$

Using this distance function, we can state our main result:

**Theorem 1.** *Let  $X^{(d)} = \{X_t^{(d)}\}_{t \geq 0}$  be a stochastic process on  $(\mathcal{X}, \mathcal{F}, \rho)$ , for each  $d \in \mathbf{N}$ , which converges weakly in the Skorokhod topology as  $d \rightarrow \infty$  to another stochastic process  $X^{(\infty)} = \{X_t^{(\infty)}\}_{t \geq 0}$ , i.e.  $X_t^{(d)} \Rightarrow X_t^{(\infty)}$  for each fixed  $t \geq 0$ . Assume these processes all have the same stationary probability distribution  $\pi$ , and that  $X^{(\infty)}$  converges (either weakly or in*

total variation distance) to  $\pi$ . Then for any  $\epsilon > 0$ , there are  $D < \infty$  and  $T < \infty$  such that

$$\mathbf{E}_{X_0^{(d)} \sim \pi} \|\mathcal{L}_{X_0^{(d)}}(X_t^{(d)}) - \pi\|_{KR} < \epsilon, \quad t \geq T, \quad d \geq D.$$

Theorem 1 may be summarised as saying that if a sequence  $\{X^{(d)}\}$  of Markov processes converges weakly to a limiting ergodic process, then we can bound the convergence of the sequence of processes uniformly over all sufficiently large  $d$ , i.e. the processes converge in  $O(1)$  iterations with respect to  $d$ . We will next apply this result to previously known diffusion limits of common MCMC algorithms.

### 3. Application to MCMC.

Our primary interest is in the use of Theorem 1 to bound the complexity of MCMC algorithms. We begin with the most popular MCMC algorithm, the Random-Walk Metropolis (RWM) algorithm. This algorithm proceeds, given a positive target probability density  $\pi_d$  on the state space  $\mathbf{R}^d$ , by running a Markov chain  $\{\mathbf{Z}_n^d\}_{n=0}^\infty$  as follows. Given the value  $\mathbf{Z}_n^d$ , a proposed new state  $\mathbf{Y}_{n+1}^d \sim MVN(\mathbf{Z}_n^d, \sigma_d^2)$  is chosen from a multivariate normal distribution centered at  $\mathbf{Z}_n^d$ , and then with probability  $\min[1, \pi(\mathbf{Y}_{n+1}^d)/\pi(\mathbf{Z}_n^d)]$  the proposal is accepted and  $\mathbf{Z}_{n+1}^d = \mathbf{Y}_{n+1}^d$ , otherwise with the remaining probability the proposal is rejected and  $\mathbf{Z}_{n+1}^d = \mathbf{Z}_n^d$ . This algorithm is easily seen to be irreducible and aperiodic and to leave  $\pi$  stationary, so it will converge asymptotically to  $\pi$ . The question then becomes how quickly it will converge, and what choice of proposal variance  $\sigma_d^2$  is optimal.

In this context, Roberts et al. (1997) proved the remarkable result that  $U^d \Rightarrow U$  as  $d \rightarrow \infty$ , where  $U_t^d = \mathbf{Z}_{[dt],1}^d$  is the first coordinate of the RWM algorithm sped up by a factor of  $d$ , and  $U$  is a limiting ergodic Langevin diffusion, and  $\Rightarrow$  indicates weak convergence in the usual Skorokhod topology. They proved this result under certain strong technical assumptions, namely that  $\pi_d$  takes on the special product form  $\pi_d(\mathbf{x}) = \prod_{i=1}^d h(x_i)$  for some fixed function  $h : \mathbf{R} \rightarrow (0, \infty)$  with  $h'/h$  Lipschitz continuous, and  $\int [h'(x)/h(x)]^8 h(x) dx < \infty$ , and  $\int [h''(x)/h(x)]^4 h(x) dx < \infty$ . They also assumed the other coordinates 2 through  $d$

of the process  $\mathbf{Z}^d$  are in stationarity, and that  $\sigma_d^2 = \ell^2/(d-1)$  for some fixed  $\ell > 0$ .

This theorem of Roberts et al. (1997) allowed them to study the limiting diffusion  $U$  as a function of the proposal variance parameter  $\ell$ , and optimise it to prove that the algorithm converges fastest when its asymptotic acceptance rate is equal to 0.234... (see also Roberts and Rosenthal, 2001). Furthermore, since their process  $U^d$  involved speeding up the original algorithm by a factor of  $d$ , their results seemed to imply that RWM required  $O(d)$  iterations to converge. However, a precise statement of such a complexity bound was not provided.

In light of Theorem 1 above, we are now able to use the diffusion limit of Roberts et al. (1997) to give an actual complexity bound on the RWM algorithm. Indeed, applying Theorem 1 to their limit immediately yields:

**Theorem 2.** *Let  $Z^{(d)}$  be a RWM algorithm on a product density in  $d$  dimensions satisfying the technical assumptions of Roberts et al. (1997). Then for any  $\epsilon > 0$ , there is  $D < \infty$  and  $T < \infty$  such that*

$$\mathbf{E}_{Z_0^{(d)} \sim \pi} \|\mathcal{L}_{Z_0^{(d)}}(Z_{[dt],1}^{(d)}) - h\|_{KR} < \epsilon, \quad t \geq T, \quad d \geq D.$$

*Hence, the RWM algorithm takes  $O(d)$  iterations to converge to within  $\epsilon$  of stationarity in any one coordinate.*

We believe this to be the first precise general result about the convergence order of the RWM algorithm. Of course, it requires the strong technical assumptions of Roberts et al. (1997), but it still applies to a fairly general collection of densities on  $\mathbf{R}^d$ . Furthermore, it appears empirically (see e.g. Roberts and Rosenthal, 2001) that even when RWM algorithms do not satisfy the technical assumptions they still exhibit similar limiting behaviour.

Another MCMC diffusion limit concerns the Metropolis-Adjusted Langevin Algorithm (MALA). This algorithm is similar to the above Random-Walk Metropolis algorithm, except that now the proposal state  $\mathbf{Y}_{n+1}^d \sim MVN(\mathbf{Z}_n^d + \frac{1}{2}\sigma_d^2 \nabla \log \pi_d(Z_n^d), \sigma_d^2)$  is chosen from a multivariate normal distribution centered at  $\mathbf{Z}_n^d + \frac{1}{2}\sigma_d^2 \nabla \log \pi_d(Z_n^d), \sigma_d^2)$  (to better approximate  $\pi$ ),

and the above acceptance probability is modified by the ratio of the corresponding proposal normal distributions. In this context, Roberts and Rosenthal (1999) proved that  $U^d \Rightarrow U$ , where  $U_t^d = \mathbf{Z}_{\lfloor d^{1/3}t \rfloor, 1}^d$  is the first coordinate of the MALA algorithm sped up by a factor of  $d^{1/3}$ , and  $U$  is again a limiting ergodic Langevin diffusion. This result again required strong technical assumptions, this time that  $\pi_d(\mathbf{x}) = \prod_{i=1}^d h(x_i)$  for some fixed function  $h : \mathbf{R} \rightarrow (0, \infty)$  with polynomially-bounded log-derivatives of all orders, and finite moments of all orders, with  $h'/h$  Lipschitz continuous. They also assumed that coordinates 2 through  $d$  of  $\mathbf{Z}^d$  are again in stationarity, and that  $\sigma_d^2 = \ell^2 d^{-1/3}$  for some fixed  $\ell > 0$ .

This theorem of Roberts and Rosenthal (1999) allowed them to optimise the limiting diffusion  $U$  as a function of  $\ell$ , and to prove that the algorithm converges fastest when its asymptotic acceptance rate is equal to 0.574... Also, since their process  $U^d$  involved speeding up the original algorithm by a factor of  $d^{1/3}$ , their results seemed to imply that MALA required  $O(d^{1/3})$  iterations to converge. Once again, we can use Theorem 1 above to obtain a more formal complexity bound:

**Theorem 3.** *Let  $Z^{(d)}$  be a MALA algorithm on a product density in  $d$  dimensions satisfying the technical assumptions of Roberts and Rosenthal (1999). Then for any  $\epsilon > 0$ , there is  $D < \infty$  and  $T < \infty$  such that*

$$\mathbf{E}_{Z_0^{(d)} \sim \pi} \|\mathcal{L}_{Z_0^{(d)}}(Z_{\lfloor d^{1/3}t \rfloor, 1}^{(d)}) - h\|_{KR} < \epsilon, \quad t \geq T, \quad d \geq D.$$

*Hence, the MALA algorithm takes  $O(d^{1/3})$  iterations to converge to within  $\epsilon$  of stationarity in any one coordinate.*

Finally, we note that a number of other diffusion limits have been proven for MCMC algorithms in other contexts. For example, Bédard (2007, 2008) and Sherlock and Roberts (2009) have extended the original RWM diffusion limit to more general target distributions; Roberts (1998) and Neal and Roberts (2006, 2008, 2011) and Jourdain et al. (2013a, 2013b) have extended it to other related cases; and Neal et al. (2012) have established diffusion

limits for RWM algorithms on discontinuous target densities. Each of these diffusion limit results could also be combined with Theorem 1 above to yield complexity order bounds in new contexts.

## 4. Proof of Theorem 1.

In this section, we prove Theorem 1. Along the way, we establish that  $\|\cdots\|_{KR}$  metrises weak convergence (Proposition 6), and that  $\mathbf{E}_{X_0 \sim \pi} \|\mathcal{L}_{X_0}(X_t^{(d)}) - \pi\|_{KR}$  is a non-increasing function of  $t$  (Lemma 11). We first establish that  $\|\cdots\|_{KR}$  is a norm:

**Lemma 4.** *Let  $S$  be any non-empty collection of functionals  $\mathcal{X} \rightarrow \mathbf{R}$  which is symmetric (i.e. if  $f \in S$  then  $-f \in S$ ). Let  $\|\mu\| = \sup_{f \in S} \mu(f)$ . Then  $\|\dots\|$  is a (possibly infinite) norm function on the set of all signed measures on  $(\mathcal{X}, \mathcal{F})$ . In particular,  $\|\cdots\|_{KR}$  is a norm.*

**Proof.** It is immediate that  $\|0\| = 0$ , and that  $\|a\mu\| = a\|\mu\|$  for  $a > 0$ . The symmetry of  $S$  implies that  $\|-\mu\| = \|\mu\|$ . Finally, for the triangle inequality, we check that

$$\|\mu + \nu\| = \sup_{f \in S} (\mu(f) + \nu(f)) \leq \left( \sup_{f \in S} \mu(f) \right) + \left( \sup_{f \in S} \nu(f) \right) = \|\mu\| + \|\nu\|.$$

Hence,  $\|\dots\|$  is a norm. The claim about  $\|\cdots\|_{KR}$  then follows by taking  $S = \text{Lip}_1^1$ . ■

We next show that truncating the metric  $\rho$  does not change  $\text{Lip}_1^1$ :

**Lemma 5.** *Let  $\rho^* = \min(2, \rho)$ . Then*

$$\text{Lip}_1^1 = \{f : \mathcal{X} \rightarrow \mathbf{R}, |f(x) - f(y)| \leq \rho^*(x, y) \ \forall x, y \in \mathcal{X}, |f| \leq 1\}.$$

**Proof.** This is immediate since we always have  $|f(x) - f(y)| \leq 2$  for  $f \in \text{Lip}_1^1$ . ■

**Proposition 6.** *The metric  $\Delta(\mu, \nu) := \|\mu - \nu\|_{KR}$  metrises weak convergence of probability measures on  $(\mathcal{X}, \mathcal{F}, \rho)$ . That is, if  $\{\mu_t\}$  and  $\mu$  are probability measures on  $(\mathcal{X}, \mathcal{F}, \rho)$ , then  $\{\mu_t\} \Rightarrow \mu$  if and only if  $\lim_{t \rightarrow \infty} \Delta(\mu_t, \mu) = 0$ .*



**Proof.** Let  $\rho^*$  be as in Lemma 5. We first note that since  $\rho$  and  $\rho^*$  agree for distances  $\leq 2$ , they give rise to precisely the same open subsets. Therefore,  $(\mathcal{X}, \rho^*)$  induces the same Borel  $\sigma$ -algebra  $\mathcal{F}$  that  $(\mathcal{X}, \rho)$  does, and thus gives rise to the same Skorokhod topology. Hence, weak convergence on  $(\mathcal{X}, \mathcal{F}, \rho)$  is precisely equivalent to weak convergence on  $(\mathcal{X}, \mathcal{F}, \rho^*)$ . Furthermore, by Lemma 5, the metric  $\|\cdots\|_{KR}$  is the same on  $(\mathcal{X}, \mathcal{F}, \rho^*)$  as on  $(\mathcal{X}, \mathcal{F}, \rho)$ . Hence, it suffices to prove the result on the truncated space  $(\mathcal{X}, \mathcal{F}, \rho^*)$ .

Now, since  $(\mathcal{X}, \mathcal{F}, \rho^*)$  is a bounded metric space, it is known (see e.g. Givens and Shortt, 1984, Proposition 4) that weak convergence on  $(\mathcal{X}, \mathcal{F}, \rho^*)$  is metrised by the Wasserstein metric  $W_1$  on  $(\mathcal{X}, \rho^*)$ , defined by

$$W_1(\mu, \nu) := \inf \mathbf{E}[\rho(X, Y)]$$

where the infimum is taken over all pairs  $(X, Y)$  of random variables on  $(\mathcal{X}, \mathcal{F})$  such that  $\mathcal{L}(X) = \mu$  and  $\mathcal{L}(Y) = \nu$ . On the other hand, again since  $(\mathcal{X}, \mathcal{F}, \rho^*)$  is a bounded metric space, it is known (Kantorovich and Rubinstein, 1958; see e.g. Givens and Shortt, 1984, p. 233) that for probability measures  $\mu$  and  $\nu$  on  $(\mathcal{X}, \mathcal{F}, \rho^*)$ , the Wasserstein metric  $W_1(\mu, \nu)$  is precisely equal to  $\|\mu - \nu\|_{KR}$ . Combining these two facts, the result follows for  $(\mathcal{X}, \mathcal{F}, \rho^*)$ , and hence also for  $(\mathcal{X}, \mathcal{F}, \rho)$ . ■

**Lemma 7.** *If  $X^{(\infty)}$  converges to  $\pi$ , either weakly or in total variation distance, then for all  $\epsilon > 0$  there is  $T < \infty$  such that  $\|\mathcal{L}_x(X_T^{(\infty)}) - \pi\|_{KR} \leq \epsilon/2$  for all  $t \geq T$ .*

**Proof.** If the convergence is weak, then this follows from Proposition 6. If the convergence is in total variation distance, then this still follows since  $\|\cdots\|_{KR} \leq \|\cdots\|_{TV}$ . ■

**Proposition 8.** *Under the assumptions of Theorem 1, for any  $x \in \mathcal{X}$  and  $\epsilon > 0$ , there is  $D < \infty$  and  $T < \infty$  such that*

$$\|\mathcal{L}_x(X_T^{(d)}) - \pi\|_{KR} < \epsilon, \quad d \geq D.$$

**Proof.** Using Lemma 4, we have by the triangle inequality that

$$\|\mathcal{L}_x(X_t^{(d)}) - \pi\|_{KR} \leq \|\mathcal{L}_x(X_t^{(d)}) - \mathcal{L}_x(X_t^{(\infty)})\|_{KR} + \|\mathcal{L}_x(X_t^{(\infty)}) - \pi\|_{KR}.$$

By Lemma 7, there is  $T < \infty$  such that  $\|\mathcal{L}_x(X_T^{(\infty)}) - \pi\|_{KR} \leq \epsilon/2$ . Then, since  $X_T^{(d)}$  converges weakly to  $X_T^{(\infty)}$ , by Proposition 6 there is  $D < \infty$  such that for all  $d \geq D$ ,  $\|\mathcal{L}_x(X_T^{(d)}) - \mathcal{L}_x(X_T^{(\infty)})\|_{KR} < \epsilon/2$ . The result follows.  $\blacksquare$

**Remark 9.** If the weak convergence of  $X^{(d)}$  to  $X^{(\infty)}$  is assumed to be uniform over bounded time intervals, then we can strengthen Proposition 8 to say that for any  $x \in \mathcal{X}$  and  $\epsilon > 0$  and  $S < \infty$ , there are  $D < \infty$  and  $T < \infty$  such that  $\|\mathcal{L}_x(X_t^{(d)}) - \pi\|_{KR} < \epsilon$  for all  $t \in [T, T + S]$ .

**Corollary 10.** Under the assumptions of Theorem 1, for any  $\epsilon > 0$ , there is  $D < \infty$  and  $T < \infty$  such that

$$\mathbf{E}_{X_0 \sim \pi} \|\mathcal{L}_{X_0}(X_T^{(d)}) - \pi\|_{KR} < \epsilon, \quad d \geq D.$$

**Proof.** We first let

$$A_m = \{x \in \mathcal{X} : \|\mathcal{L}_x(X_t^{(d)}) - \pi\|_{KR} < \epsilon/2 \quad \forall t \geq m\}.$$

Then  $A_{m+1} \subseteq A_m$  by inspection, and  $\bigcup_m A_m = \mathcal{X}$  by Lemma 7. Hence, by continuity of probabilities (see e.g. Proposition 3.3.1 of Rosenthal, 2000),  $\lim_{m \rightarrow \infty} \pi(A_m) = 1$ . We can therefore find  $T < \infty$  such that  $\pi(A_T) \geq 1 - (\epsilon/8)$ .

Next, for this fixed  $T$ , let

$$B_m = \{x \in \mathcal{X} : \|\mathcal{L}_x(X_T^{(d)}) - \pi\|_{KR} < \epsilon/2 \quad \forall d \geq m\}.$$

Then  $B_{m+1} \subseteq B_m$  by inspection, and  $\bigcup_m B_m = \mathcal{X}$  by Proposition 8, so again by continuity of probabilities we can find  $D \in \mathbf{N}$  such that  $\pi(B_D) \geq 1 - (\epsilon/8)$ .

We then compute that for this fixed  $T$  and  $D$ , and for any  $d \geq D$ ,

$$\begin{aligned}
& \mathbf{E}_{X_0 \sim \pi} \|\mathcal{L}_{X_0}(X_T^{(d)}) - \pi\|_{KR} \\
&= \mathbf{E}_{X_0 \sim \pi} \left( \mathbf{1}_{X_0 \in A_T \cap B_D} \|\mathcal{L}_{X_0}(X_T^{(d)}) - \pi\|_{KR} \right) + \mathbf{E}_{X_0 \sim \pi} \left( \mathbf{1}_{X_0 \notin A_T \cap B_D} \|\mathcal{L}_{X_0}(X_T^{(d)}) - \pi\|_{KR} \right) \\
&\leq (\epsilon/2) + [(\epsilon/8) + (\epsilon/8)] \times 2 = \epsilon,
\end{aligned}$$

where we have used the fact that by definition we always have  $\|\mathcal{L}_x(X_T^{(d)}) - \pi\|_{KR} \leq 2$  for any  $x$  and  $d$ . This gives the result.  $\blacksquare$

Corollary 10 is nearly what we need to prove Theorem 1. However, for Theorem 1 we want the convergence to be within  $\epsilon$  for *all*  $t \geq T$ , not just for one fixed  $T$  (nor just for all  $t$  in some bounded time interval, cf. Remark 9). Unfortunately,  $\|\mathcal{L}_x(X_t^{(d)}) - \pi\|_{KR}$  might not be a non-increasing function of  $t$  (unlike  $\|\mathcal{L}_x(X_t^{(d)}) - \pi\|_{TV}$ , which always is, see e.g. Proposition 3(c) of Roberts and Rosenthal, 2004). On the other hand, fortunately the quantity  $\mathbf{E}_{X_0 \sim \pi} \|\mathcal{L}_{X_0}(X_t^{(d)}) - \pi\|_{KR}$  is indeed non-increasing:

**Lemma 11.** *Let  $\|\dots\|$  be any norm function on signed measures on  $(\mathcal{X}, \mathcal{F})$ . Let  $P^t(x, \cdot)$  be the transition probabilities for a Markov chain on  $(\mathcal{X}, \mathcal{F})$  with stationary probability distribution  $\pi$ . Let  $\text{dist}(t) = \mathbf{E}_{X_0 \sim \pi} \|P^t(X_0, \cdot) - \pi\|$ . Then  $\text{dist}(t)$  is a non-increasing function of  $t$ . In particular, in the context of Theorem 1,  $\mathbf{E}_{X_0 \sim \pi} \|\mathcal{L}_{X_0}(X_t^{(d)}) - \pi\|_{KR}$  is a non-increasing function of  $t$ .*

**Proof.** We compute by stationarity that for  $s, t > 0$ ,

$$\begin{aligned}
\text{dist}(s+t) &= \mathbf{E}_{X_0 \sim \pi} \|P^{s+t}(X_0, \cdot) - \pi\| \\
&= \mathbf{E}_{X_0 \sim \pi} \left\| \int_{y \in \mathcal{X}} P^s(X_0, dy) P^t(y, \cdot) - \pi \right\| \\
&\leq \mathbf{E}_{X_0 \sim \pi} \int_{y \in \mathcal{X}} P^s(X_0, dy) \|P^t(y, \cdot) - \pi\|
\end{aligned}$$

$$= \mathbf{E}_{Y_0 \sim \pi} \|P^t(Y_0, \cdot) - \pi\| = \text{dist}(t),$$

thus proving the first claim. The claim about  $\mathbf{E}_{x \sim \pi} \|\mathcal{L}_x(X_t^{(d)}) - \pi\|_{KR}$  then follows by Lemma 4 upon setting  $P^t(x, A) = \mathbf{P}[X_t^{(d)} \in A \mid X_0^{(d)} = x]$ . ■

Theorem 1 then follows by combining Corollary 10 and Lemma 11.

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